Final Exam — Analysis (WPMA14004)

Thursday 16 June 2016, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (3 + 12 points)

(a) State the Axiom of Completeness.

(b) Assume that the sets $A, B \subset \mathbb{R}$ are both bounded above. Prove that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

Hint: first explain that it suffices to consider only the case $\sup A \leq \sup B$.

Problem 2 (4 + 4 + 7 points)

Consider the sequences (t_k) and (s_n) given by

$$t_k = \frac{1}{k} - \ln\left(\frac{k+1}{k}\right) \qquad \text{and} \qquad s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1).$$

Prove the following statements:

(a) $\displaystyle\sum_{k=1}^{n} t_k = s_n$ for all $n \in \mathbb{N}$.

(b) $0 \leq t_k \leq \dfrac{1}{2k^2}$ for all $k \in \mathbb{N}$. Hint: $x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x$ for all $x \geq 0$.

(c) (s_n) is convergent.

Problem 3 (5 + 10 points)

Let $B \subset \mathbb{R}$ be a set of positive real numbers with the following "finite sum property": adding finitely many elements of B gives a sum of 1 or less.

Prove the following statements:

(a) For all $\epsilon > 0$ there exist only finitely many $x \in B$ with $x > \epsilon$.

(b) $B \cup \{0\}$ is compact.

Problem 4 (4 + 4 + 7 points)

Consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{x}{1 + |x|}.$$

Prove the following statements:

(a) f is differentiable at $x = 0$ and $f'(0) = 1$.

(b) f is differentiable at $x \neq 0$ and $0 < f'(x) < 1$.

(c) f is uniformly continuous on $\mathbb{R}$.

Problem 5 (3 + 6 + 6 points)

Let $g : \mathbb{R} \to \mathbb{R}$ be a function with domain $\mathbb{R}$. Consider the following sequence:

$$f_n(x) = \frac{ng(x)}{n + |g(x)|}.$$

Prove the following statements:

(a) $|f_n(x) - g(x)| \leq \dfrac{g(x)^2}{n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(b) If g is bounded on $\mathbb{R}$, then $f_n \to g$ uniformly on $\mathbb{R}$.

(c) If g is continuous on $\mathbb{R}$, then $f_n \to g$ uniformly on all compact subsets of $\mathbb{R}$.

Problem 6 (9 + 6 points)

Consider the modified Dirichlet function $h : [0, 1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Show that $U(h, P) > \frac{1}{2}$ for any partition P of $[0, 1]$.

Hint: prove that $x_k(x_k - x_{k-1}) > \frac{1}{2}(x_k + x_{k-1})(x_k - x_{k-1})$.

(b) Is h integrable on $[0, 1]$?

End of test (90 points)

Solution of Problem 1 (3 + 12 points)

(a) Every nonempty set of real numbers that is bounded above has a least upper bound. **(3 points)**

(b) Without loss of generality we may assume that $\sup A \leq \sup B$. Otherwise we just exchange the names of the sets A and B.

An alternative argument is that the case $\sup B \leq \sup A$ has a similar proof since the set A and B appear in the formula in a symmetric way (i.e., interchanging the roles of A and B gives the same formula).
(4 points)

Therefore, we need to prove that $\sup(A \cup B) = \sup B$. To that end, we need to prove two things:

(i) $\sup B$ is an upper bound for $A \cup B$;

(ii) any other upper bound u of $A \cup B$ satisfies $\sup B \leq u$ or any number smaller than $\sup B$ is no longer an upper bound of $A \cup B$.

Let $x \in A \cup B$ be arbitrary, then either $x \in A$ or $x \in B$. Therefore, either $x \leq \sup A$ or $x \leq \sup B$. Since $\sup A \leq \sup B$ it follows that $x \leq \sup B$ for all $x \in A \cup B$. We conclude that $\sup B$ is an upper bound for the set $A \cup B$.
(4 points)

Let u be any upper bound for $A \cup B$. Since $x \leq u$ for all $x \in A \cup B$ it follows in particular that $x \leq u$ for all $x \in B$. Since $\sup B$ is the least upper bound of B it follows that $\sup B \leq u$ which also shows that $\sup B$ is the least upper bound of $A \cup B$.
(4 points)

Alternative argument. Let $\epsilon > 0$ be arbitrary then there exists an element $x \in B$ such that $\sup B - \epsilon < x$. This means that $\sup B - \epsilon$ is not an upper bound for B. Since $B \subset A \cup B$ it follows that $\sup B - \epsilon$ cannot be an upper bound for $A \cup B$. We conclude that $\sup B$ is the least upper bound of $A \cup B$.
(4 points)

Solution of Problem 2 (4 + 4 + 7 points)

(a) For all $n \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{k=1}^{n} t_k &= \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n}(\ln(k) - \ln(k+1)) \\ &= \sum_{k=1}^{n} \frac{1}{k} + \ln(1) - \ln(n+1) \\ &= \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) = s_n\end{aligned}$$

where we have used the telescoping property of the second sum.
(4 points)

(b) Using the hint with $x = 1/k$ gives

$$\frac{1}{k} - \frac{1}{2k^2} \leq \ln\left(1 + \frac{1}{k}\right) \leq \frac{1}{k}$$

or, equivalently,

$$-\frac{1}{2k^2} \leq \ln\left(1 + \frac{1}{k}\right) - \frac{1}{k} \leq 0.$$

Multiplying these inequalities by -1 gives the desired result.
(4 points)

(c) The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (standard result on infinite series) and therefore $\sum_{k=1}^{\infty} \frac{1}{2k^2}$ converges as well.
(3 points)

By the comparison test the series $\sum_{k=1}^{\infty} t_k$ converges.
(3 points)

Since s_n is precisely the n-th partial sum of the series $\sum_{k=1}^{\infty} t_k$ it follows that the sequence (s_n) converges.
(1 point)

Solution of Problem 3 (5 + 10 points)

(a) Let $\epsilon > 0$ be arbitrary. If there exist infinitely many $x \in B$ such that $x > \epsilon$, then we can certainly pick N of them with $N > 1/\epsilon$. Hence their sum will be larger than $N\epsilon = 1$, which contradicts the finite-sum property of B.
(5 points)

(b) *Proof using open covers.* Let $\{O_i : i \in I\}$ be an open cover for $B \cup \{0\}$. Pick $i_0 \in I$ such that $0 \in O_{i_0}$. Since O_{i_0} is open there exists $\epsilon > 0$ such that $V_\epsilon(0) \subset O_{i_0}$.
(4 points)

From part (a) it follows that only finitely many points $x_1, \ldots, x_n$ of B are not included in $V_\epsilon(0)$. Since $\{O_i : i \in I\}$ covers $B \cup \{0\}$ there exist indices $i_k \in I$ such that $x_k \in O_{i_k}$ for $k = 1, \ldots, n$.
(4 points)

We conclude that $B \cup \{0\} \subset O_{i_0} \cup O_{i_1} \cup \cdots \cup O_{i_n}$. This proves that every open cover of $B \cup \{0\}$ has a finite subcover, and hence proves that $B \cup \{0\}$ is a compact set.
(2 points)

Proof via closed and bounded. Applying the definition of the finite-sum property to a set with one element shows that each $x \in B$ satisfies $0 < x < 1$. Therefore, B, and hence $B \cup \{0\}$, is a bounded set.
(4 points)

Note that if $x < 0$, then x is *not* a limit point of B. If $x \in B$ then part (a) shows that x is isolated. Hence, the only possible limit point of B is the point $x = 0$. This shows that $B \cup \{0\} = \overline{B}$ is closed.
(4 points)

We conclude that $B \cup \{0\}$ is a closed and bounded set and hence compact.
(2 points)

Solution of Problem 4 (4 + 4 + 7 points)

(a) We compute $f'(0)$ by means of the definition:

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{1}{1+|x|} = 1.$$

(4 points)

Alternative proof. Note that

$$\left| \frac{f(x) - f(0)}{x - 0} - 1 \right| = \left| \frac{1}{1+|x|} - 1 \right| = \frac{|x|}{1+|x|} \leq |x|.$$

Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon$ then

$$0 < |x - 0| < \delta \quad \Rightarrow \quad \left| \frac{f(x) - f(0)}{x - 0} - 1 \right| < \epsilon$$

which shows that f is differentiable at $x = 0$ and $f'(0) = 1$.
(4 points)

(b) On the interval $(0, \infty)$ we have $|x| = x$, which implies that f is differentiable and the usual rules from calculus may be applied:

$$f'(x) = \frac{(1+x) - x}{(1+x)^2} = \frac{1}{(1+|x|)^2},$$

which also shows that $0 < f'(x) < 1$.
(2 points)

On the interval $(-\infty, 0)$ we have $|x| = -x$, which implies that f is differentiable and the usual rules from calculus may be applied:

$$f'(x) = \frac{(1-x) + x}{(1-x)^2} = \frac{1}{(1+|x|)^2},$$

which also shows that $0 < f'(x) < 1$.
(2 points)

(c) Since f is differentiable on $\mathbb{R}$ we may use the Mean Value theorem. Let $x < y$, then there exists a point $c \in (x, y)$ such that

$$f(x) - f(y) = f'(c)(x - y).$$

(3 points)

Taking absolute values and using parts (a) and (b) gives

$$|f(x) - f(y)| = |f'(c)|\,|x - y| \leq |x - y|.$$

(2 points)

Let $\epsilon > 0$ be arbitrary and take $\delta = \epsilon$ then

$$|x - y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \delta = \epsilon,$$

for all $x, y \in \mathbb{R}$, which shows that f is uniformly continuous on $\mathbb{R}$.
(2 points)

Solution of Problem 5 (3 + 6 + 6 points)

(a) For all $n \in \mathbb{N}$ and $x \in \mathbb{R}$ we have

$$\begin{aligned} |f_n(x) - g(x)| &= \left| \frac{ng(x)}{n + |g(x)|} - g(x) \right| \\ &= \left| \frac{ng(x)}{n + |g(x)|} - \frac{(n + |g(x)|)g(x)}{n + |g(x)|} \right| \\ &= \frac{|g(x)|^2}{n + |g(x)|} \\ &\leq \frac{|g(x)|^2}{n}. \end{aligned}$$

(3 points)

(b) If g is bounded, then there exists a constant $C > 0$ such that $|g(x)| \leq C$ for all $x \in \mathbb{R}$. Therefore, part (a) implies that

$$|f_n(x) - g(x)| \leq \frac{C^2}{n} \qquad \text{for all } x \in \mathbb{R} \text{ and } n \in \mathbb{N}.$$

(2 points)

Let $\epsilon > 0$ be arbitrary, and pick $N \in \mathbb{N}$ such that $N > C^2/\epsilon$, then

$$n \geq N \quad \Rightarrow \quad |f_n(x) - g(x)| \leq \frac{C^2}{N} < \epsilon \qquad \text{for all } x \in \mathbb{R}.$$

We conclude that $f_n \to g$ uniformly on $\mathbb{R}$.
(4 points)

(c) Now assume that g is continuous on $\mathbb{R}$. If $K \subset \mathbb{R}$ is compact, then g attains a maximum and minimum value on K, i.e., there exist $a, b \in K$ such that

$$g(a) \leq g(x) \leq g(b) \qquad \text{for all } x \in K.$$

(3 points)

In particular, it follows that g is bounded on K:

$$|g(x)| \leq C = \max\{|g(a)|, |g(b)|\} \qquad \text{for all } x \in K.$$

(2 points)

By repeating the argument of part (b), with $\mathbb{R}$ replaced by K, it follows that $f_n \to g$ uniformly on K.
(1 point)

Problem 6 (9 + 6 points)

(a) Let P be any partition of $[0,1]$. Note that the supremum of h over each subinterval in P is given by

$$M_k = \sup\{h(x) \,:\, x \in [x_{k-1}, x_k]\} = x_k, \qquad k = 1, \ldots, n.$$

Therefore, the upper sum of h with respect to P is given by

$$U(h,P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} x_k(x_k - x_{k-1}).$$

(3 points)

Note that for all $k = 1, \ldots, n$ we have

$$x_k > x_{k-1} \quad \Rightarrow \quad x_k + x_k > x_k + x_{k-1} \quad \Rightarrow \quad x_k > \tfrac{1}{2}(x_k + x_{k-1}).$$

(3 points)

Therefore, we obtain the following lower bound for the upper sum:

$$\begin{aligned} U(h,P) &> \sum_{k=1}^{n} \tfrac{1}{2}(x_k + x_{k-1})(x_k - x_{k-1}) \\ &= \tfrac{1}{2}\sum_{k=1}^{n}(x_k^2 - x_{k-1}^2) \\ &= \tfrac{1}{2}(x_n^2 - x_0^2) = \tfrac{1}{2}. \end{aligned}$$

(3 points)

(b) Let P be any partition of $[0,1]$. Note that the infimum of h over each subinterval in P is given by

$$m_k = \inf\{h(x) \,:\, x \in [x_{k-1}, x_k]\} = 0, \qquad k = 1, \ldots, n.$$

Therefore, the lower sum of h with respect to P is given by $L(h,P) = 0$.
(3 points)

Combing this result with part (a) shows that for all partitions P of $[0,1]$ we have

$$U(h,P) - L(h,P) > \tfrac{1}{2}.$$

We conclude that h is not integrable on $[0,1]$.
(3 points)